

# THE $\bar{\partial}$ -CAUCHY PROBLEM AND NONEXISTENCE OF LIPSCHITZ LEVI-FLAT HYPERSURFACES IN $\mathbb{C}P^n$ WITH $n \geq 3$

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In this paper we study the  $\bar{\partial}$ -Cauchy problem and the  $\bar{\partial}$ -closed extension problem for forms on domains in complex hermitian manifolds. These problems were first studied in the paper by Kohn-Rossi [KR] (see also [FK]), who proved the holomorphic extension of smooth  $CR$  functions and the  $\bar{\partial}$ -closed extension of smooth forms from the boundary  $b\Omega$  of a strongly pseudoconvex domain to the whole domain  $\Omega$ . The  $L^2$  theory of these problem has been obtained for pseudoconvex domains in  $\mathbb{C}^n$  or, more generally, for domains in complex manifolds with strongly plurisubharmonic weight functions (see Chapter 9 in [CS] and the references therein). In this paper we study these problems on pseudoconvex domains in complex hermitian manifolds when such weight functions are not available, for instance, on a pseudoconvex domain in the complex projective space  $\mathbb{C}P^n$ .

One application of the  $\bar{\partial}$ -Cauchy problem is to obtain the nonexistence of Levi-flat hypersurfaces in  $\mathbb{C}P^n$ . This was first used by Siu in [Si1] where the nonexistence of smooth (or  $\frac{3n}{2} + 7$ ) Levi-flat hypersurfaces in  $\mathbb{C}P^n$  was proved for  $n \geq 3$ . In a subsequent paper [Si2], he proved the nonexistence of  $C^8$  Levi-flat hypersurfaces in  $\mathbb{C}P^2$ . We also mention the papers by Lins-Neto [LN], Iordan [Io] and Ni-Wolfson [NW] on related subjects.

The main result of this paper is to prove the nonexistence of Lipschitz Levi-flat hypersurfaces in  $\mathbb{C}P^n$  for  $n \geq 3$ . We first define Lipschitz Levi-flat hypersurfaces.

Recall that a bounded domain  $D \subset \subset \mathbb{R}^{2n}$  is called Lipschitz if near every boundary point  $p \in bD$ , there exists a neighborhood  $U$  of  $p$  such that in local coordinates  $(x', x_n) = (x_1, \dots, x_{2n-1}, x_{2n})$ ,

$$D \cap U = \{(x', x_{2n}) \in U \mid x_{2n} > \psi(x')\}$$

for some Lipschitz function  $\psi : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}$ . A Lipschitz function is differentiable almost everywhere (See Evans-Gariepy [EG] for a proof of this fact). A domain in a complex manifold is called Lipschitz if at every point of the boundary, there exist some local coordinates such that the boundary is the graph of some Lipschitz function.

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**Definition.** A Lipschitz hypersurface is a hypersurface which locally is the graph of a Lipschitz function. A Lipschitz (or  $C^1$ ) hypersurface is said to be Levi-flat if it is locally foliated by complex manifolds of complex dimension  $n - 1$ .

From the implicit function theorem, any  $C^1$  hypersurface locally is the graph of some  $C^1$  function. A  $C^2$  hypersurface  $M$  is called Levi-flat if its Levi-form vanishes on  $M$ . Any  $C^k$  Levi-flat hypersurface,  $k \geq 2$  is locally foliated by complex manifolds of complex dimension  $n - 1$ . The foliation is of class  $C^k$  if the hypersurface is of class  $C^k$ ,  $k \geq 2$  (see Barrett-Fornaess [BF]). The proof in [BF] also gives that if a real  $C^1$  hypersurface admits a continuous foliation by complex manifolds, then the foliation is actually  $C^1$ . Thus our definition is a natural generalization of Levi-flatness to Lipschitz or  $C^1$  hypersurfaces.

**Theorem.** *There exist no Lipschitz Levi-flat hypersurfaces in  $\mathbb{C}P^n$  for  $n \geq 3$ .*

The main tool to prove the theorem is to study the  $\bar{\partial}$ -Cauchy problem using the  $\bar{\partial}$ -Neumann operator. When the boundary is  $C^2$  and pseudoconvex in  $\mathbb{C}P^n$ , the  $\bar{\partial}$ -Neumann operator exists using bounded plurisubharmonic functions, a result by Ohsawa-Sibony [OS]. It is not known if the  $\bar{\partial}$ -Neumann operator exists for Lipschitz pseudoconvex domains. However, the weighted  $\bar{\partial}$ -Neumann operator always exists with suitable weight functions. To prove the nonexistence of Lipschitz Levi-flat hypersurfaces, we use the  $L^2$   $\bar{\partial}$ -Cauchy problem with weights and the equivalence of the weighted spaces with the Sobolev spaces.

In [CSW], we carried out an  $L^2$  approach for  $\bar{\partial}$ -closed extension problem using the  $\bar{\partial}$ -Neumann operator in order to study the nonexistence of  $C^2$ -smooth Levi-flat real hypersurfaces in  $\mathbb{C}P^n$ . In fact, only the nonexistence of  $C^{2,\alpha}$  Levi-flat hypersurfaces in  $\mathbb{C}P^n$  with  $n \geq 3$  was proved, by using  $\bar{\partial}$ -closed extension of the Chern connection  $(0, 1)$ -forms (see Section 5 in [CSW]). The proof for the  $\mathbb{C}P^2$  case in Section 6 of [CSW] relies on a Liouville-type result, which is yet to be completed (see Conjecture 2 at the end of this paper). At the end of the paper, we mention how to bridge the gap in the proof [CSW] for the nonexistence of  $C^2$  Levi-flat hypersurfaces in  $\mathbb{C}P^2$  using results in [Si2].

We note that there exist nonsmooth Levi-flat hypersurfaces in  $\mathbb{C}P^n$  which are not locally Lipschitz graphs. Let  $M = \{[z_0, z_1, z_2] \in \mathbb{C}P^2 \mid |z_0| = |z_1|\}$  and  $\mathbb{C}P^2 \setminus M = \Omega^+ \cup \Omega^-$ , where  $[z_0, z_1, z_2]$  are homogeneous coordinates in  $\mathbb{C}P^2$ . Then  $\Omega^+$  and  $\Omega^-$  are both pseudoconcave and pseudoconvex domains since each can be represented in local coordinates by a product of a disc with  $\mathbb{C}$  (see e.g. [HI]). We can view  $M$  as a Levi-flat hypersurface in the sense that it is the boundary of a domain which is both pseudoconvex and pseudoconcave. The boundary  $M$  is smooth except at  $[0, 0, 1]$ , where  $M$  is not foliated by complex curves. Notice that  $M$  is also not a graph of a Lipschitz function in a neighborhood of the point  $[0, 0, 1]$ . Similar examples can be found in  $\mathbb{C}P^n$  for  $n \geq 3$  by setting  $M = \{[z_0, z_1, z_2, \dots, z_n] \in \mathbb{C}P^n \mid |z_0| = |z_1|\}$ .

The plan of this paper is as follows: In section 1 we give a self-contained treatment of the  $\bar{\partial}$ -Cauchy problem on domains with Lipschitz boundary in a hermitian

complex manifold using the  $\bar{\partial}$ -Neumann operators. In section 2 we prove the existence of Hölder continuous bounded exhaustion functions for pseudoconvex domains with  $C^{1,1}$  boundary in  $\mathbb{C}P^n$ . This gives an alternative proof of the Ohsawa-Sibony result on the existence of bounded plurisubharmonic functions for  $C^2$  pseudoconvex domains in  $\mathbb{C}P^n$ . In Section 3, we use the weighted  $\bar{\partial}$ -Cauchy problem to study the extension of  $\bar{\partial}$ -closed  $(p, q)$ -forms from a pseudoconcave domain to  $\mathbb{C}P^n$  when  $q < n - 1$ ,  $n \geq 3$ . In Section 4, we study the Levi-flat boundary and its connection forms and prove the main theorem. It is still unknown if our main theorem can be extended to  $\mathbb{C}P^2$ . In Section 5, we discuss the extension of  $\bar{\partial}$ -closed  $(p, n - 1)$ -forms in  $\mathbb{C}P^n$ . We also mention two open problems which will imply the nonexistence of Lipschitz Levi-flat hypersurfaces in  $\mathbb{C}P^2$ .

### 1. The $L^2$ $\bar{\partial}$ -Cauchy problem on complex manifolds

Let  $\mathcal{X}$  be a complex hermitian manifold of dimension  $n \geq 2$  and let  $\Omega$  be a bounded domain in  $\mathcal{X}$ . The  $L^2$  Cauchy problem for  $\bar{\partial}$  is to study the following question: Given a  $(p, q)$ -form  $f$  with  $L^2$  coefficients supported in  $\bar{\Omega}$ , where  $0 \leq p \leq n$  and  $1 \leq q \leq n$ , find a  $(p, q - 1)$ -form  $u$  such that

$$\begin{cases} \text{Supp } u \subset \bar{\Omega}, \\ \bar{\partial}u = f \quad \text{in } \mathcal{X} \text{ in the distribution sense.} \end{cases} \quad (1.0)$$

When  $q < n$ , we assume that  $f$  satisfies

$$\bar{\partial}f = 0 \quad \text{in } \mathcal{X} \text{ in the distribution sense.} \quad (1.1)$$

When  $q = n$ , (1.1) is a void condition. Using integration-by-parts, another compatibility condition for (1.0) can be derived as follows: If (1.0) is solvable for  $f \in L^2_{(p,q)}(\Omega)$ , where  $1 \leq q \leq n$ , then  $f$  must satisfy

$$\int_{\Omega} f \wedge g = 0, \quad g \in L^2_{(n-p,n-q)}(\Omega) \cap \text{Ker}(\bar{\partial}). \quad (1.2)$$

We define the generalized Bergman projection operator

$$P_{(p,q)} : L^2_{(p,q)}(\Omega) \rightarrow L^2_{(p,q)}(\Omega) \cap \text{Ker}(\bar{\partial}).$$

Recall that the Hodge star operator  $\star = \bar{\star}$  is given by

$$(\star f, g)|_{\Omega} = (-1)^{p+q} \overline{\int_{\Omega} g \wedge f} = \overline{\int_{\Omega} f \wedge g}.$$

Hence, condition (1.2) is equivalent to

$$P_{(n-p,n-q)}(\star f) = 0. \quad (1.2')$$

Thus when  $q < n$ , both (1.1) and (1.2) are compatibility conditions for the  $\bar{\partial}$ -Cauchy problem.

In the next lemma, we will show that condition (1.2) implies condition (1.1).

**Lemma 1.1.** *Let  $\Omega$  be a bounded domain in a complex hermitian manifold  $\mathcal{X}$  of dimension  $n \geq 2$ . Let  $f \in L^2_{(p,q)}(\Omega)$ , where  $0 \leq p \leq n$  and  $0 \leq q < n$ , such that  $f$  satisfies (1.2). Then  $\bar{\partial}f = 0$  in  $\mathcal{X}$  if  $f$  is extended to be zero outside  $\Omega$ .*

*Proof.* We take  $g = \bar{\partial} \star v$  for some  $v \in C^\infty_{(p,q+1)}(\mathcal{X})$  in (1.2). It is clear that  $g \in \text{Ker}(\bar{\partial})$ . Let  $\vartheta = \bar{\partial}^* = -\star \bar{\partial} \star$ , where  $\vartheta$  is the formal adjoint of  $\bar{\partial}$  and  $\bar{\partial}^*$  is the Hilbert space adjoint. By (1.2) and the fact  $\bar{\partial} \star v \in \text{Ker}(\bar{\partial})$ , we see that

$$(f, \bar{\partial}^* v)_{\mathcal{X}} = \int_{\Omega} f \wedge \star(\bar{\partial}^* v) = (-1)^{p+q+1} \int_{\Omega} f \wedge \bar{\partial}(\star v) = 0$$

for any  $v \in C^\infty_{(p,q+1)}(\mathcal{X})$ , where we used the equality  $\star(\star u) = (-1)^{p+q}u$  for  $u = \bar{\partial}(\star v) \in L^2_{(n-p,n-q)}(\Omega)$ . This implies that  $\bar{\partial}f = 0$  in the distribution sense in  $\mathcal{X}$ .  $\square$

In general, (1.1) and (1.2) are not equivalent. We will see that they are equivalent for  $q < n$  in Theorem 1.4.

When  $q \leq n$ , including the top degree case, the  $\bar{\partial}$ -Cauchy problem will be solved for forms satisfying (1.2) in the next theorem.

**Theorem 1.2.** *Let  $\Omega$  be a bounded domain in a complex hermitian manifold  $\mathcal{X}$  of dimension  $n \geq 2$ . Suppose that the  $\bar{\partial}$ -Neumann operator  $N_{(n-p,n-q)}$  on  $L^2_{(n-p,n-q)}(\Omega)$  exists for some  $0 \leq p \leq n$  and  $1 \leq q \leq n$ . For any  $f \in L^2_{(p,q)}(\mathcal{X})$  such that  $f$  is supported in  $\bar{\Omega}$  and  $f$  satisfies (1.2), then there exists  $u \in L^2_{(p,q-1)}(\mathcal{X})$  satisfies  $\bar{\partial}u = f$  in the distribution sense in  $\mathcal{X}$  with  $u$  supported in  $\bar{\Omega}$ .*

*Proof.* Since the  $\bar{\partial}$ -Neumann operators  $N_{(n-p,n-q)}$  in  $\Omega$  exists, the generalized Bergman projection operator  $P_{(n-p,n-q)} : L^2_{(n-p,n-q)}(\Omega) \rightarrow L^2_{(n-p,n-q)}(\Omega) \cap \text{Ker}(\bar{\partial})$  is given by

$$\bar{\partial}^* \bar{\partial} N_{(n-p,n-q)} = I - P_{(n-p,n-q)}. \quad (1.3)$$

We set  $u$  by

$$u = -\star \bar{\partial} N_{(n-p,n-q)} \star f. \quad (1.4)$$

Since  $f$  satisfies (1.2), we have  $P_{(n-p,n-q)} \star f = 0$ . From (1.3), we have

$$\begin{aligned} \bar{\partial}u &= (-1)^{p+q} \star \bar{\partial}^* \bar{\partial} N_{(n-p,n-q)} \star f \\ &= f - (-1)^{p+q} \star P_{(n-p,n-q)} \star f = f \quad \text{in } \Omega. \end{aligned} \quad (1.5)$$

Using the fact that  $\star u \in \text{Dom}(\bar{\partial}^*)$  and extending  $u$  to be zero outside  $\Omega$ , one can show that  $\bar{\partial}u = f$  in  $\mathcal{X}$  in the distribution sense as follows. Observe that

$$\bar{\partial}^*(\star u) = \vartheta \star u = (-1)^{p+q} \star \bar{\partial}u = (-1)^{p+q} \star f,$$

where  $\vartheta \star u$  is taken in the distribution sense in  $\Omega$ . Hence, we have for any  $\psi \in C_{(p,q)}^\infty(\mathcal{X})$ ,

$$\begin{aligned}
(u, \vartheta\psi)_{\mathcal{X}} &= (\star\vartheta\psi, \star u)_{\Omega} \\
&= (-1)^{p+q}(\bar{\partial} \star \psi, \star u)_{\Omega} \\
&= (-1)^{p+q}(\star\psi, \bar{\partial}^*(\star u))_{\Omega} \\
&= (\star\psi, \star f)_{\Omega} \\
&= (f, \psi)_{\mathcal{X}},
\end{aligned} \tag{1.6}$$

where the third equality holds since  $\star u \in \text{Dom}(\bar{\partial}^*)$ . Thus  $\bar{\partial}u = f$  in the distribution sense in  $\mathcal{X}$ .  $\square$

Theorem 1.2 implies that condition (1.2) is necessary and sufficient for solving the  $\bar{\partial}$ -Cauchy problem for all  $(p, q)$ -forms of all degrees, including the top degree  $q = n$ .

Next we analyze the case when  $q < n$ . Let  $\mathcal{H}_{(p,q)}(\Omega)$  denote the space of harmonic  $(p, q)$ -forms, i.e.,

$$\mathcal{H}_{(p,q)}(\Omega) = \{h \in L_{(p,q)}^2(\Omega) \cap \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}^*) \mid \bar{\partial}h = 0, \bar{\partial}^*h = 0\}.$$

Notice that no assumption on the smoothness of  $\Omega$  is used in Lemma 1.1 and Theorem 1.2. From now on, we will assume that the domain  $\Omega$  has Lipschitz boundary.

**Lemma 1.3.** *Let  $\mathcal{X}$  be a complex hermitian manifold of dimension  $n \geq 2$ . Let  $\Omega$  be a bounded domain in  $\mathcal{X}$  with Lipschitz boundary. For  $0 \leq p \leq n$ ,  $1 \leq q \leq n-1$ , if  $f \in L_{(p,q)}^2(\mathcal{X})$  with  $\bar{\partial}f = 0$  in the distribution sense in  $\mathcal{X}$  and  $f$  supported in  $\bar{\Omega}$ , then  $\star f \in \text{Dom}(\bar{\partial}^*)$  and  $\bar{\partial}^* \star f = 0$  in  $\Omega$ .*

*Proof.* For any  $\phi \in C_{(n-p, n-q-1)}^\infty(\bar{\Omega})$ ,

$$\begin{aligned}
(\bar{\partial}\phi, \star f)_{\Omega} &= (-1)^{p+q} \int_{\Omega} \bar{\partial}\phi \wedge f = (-1)^{p+q} \int_{\Omega} f \wedge \star \star \bar{\partial}\phi \\
&= (-1)^{p+q} (f, \star \bar{\partial}\phi)_{\Omega} = (f, \vartheta \star \phi)_{\Omega} \\
&= (\bar{\partial}f, \star \phi)_{\mathcal{X}} \\
&= 0
\end{aligned}$$

since  $\text{supp } f \subset \bar{\Omega}$  and  $\bar{\partial}f = 0$  in the distribution sense in  $\mathcal{X}$ .

Since  $\Omega$  has Lipschitz boundary  $b\Omega$ , using the Friedrichs's lemma, we see that the set  $C_{(n-p, n-q-1)}^\infty(\bar{\Omega})$  is dense in  $\text{Dom}(\bar{\partial})$  in the graph norm (see [Hö1] or Step 1 in Lemma 4.3.2 in [CS]). It follows from the definition of  $\bar{\partial}^*$  that  $\star f \in \text{Dom}(\bar{\partial}^*)$  and  $\bar{\partial}^*(\star f) = 0$ .  $\square$

We summarize the discussion above as follows.

**Theorem 1.4.** *Let  $\mathcal{X}$  be a complex hermitian manifold of dimension  $n \geq 2$ . Let  $\Omega$  be a bounded domain in  $\mathcal{X}$  with Lipschitz boundary. We assume that the  $\bar{\partial}$ -Neumann operators  $N_{(n-p, n-q)}$  and  $N_{(n-p, n-q-1)}$  in  $\Omega$  exist for  $0 \leq p \leq n$  and  $1 \leq q \leq n-1$  and assume that  $\mathcal{H}_{(n-p, n-q)}(\Omega) = \{0\}$ . For every  $f \in L^2_{(p, q)}(\mathcal{X})$  with  $\bar{\partial}f = 0$  in the distribution sense in  $\mathcal{X}$  and  $f$  supported in  $\bar{\Omega}$ , one can find  $u \in L^2_{(p, q-1)}(\mathcal{X})$  such that  $\bar{\partial}u = f$  in the distribution sense in  $\mathcal{X}$  with  $u$  supported in  $\bar{\Omega}$ .*

*Proof.* By our assumption, the  $\bar{\partial}$ -Neumann operator  $N_{(n-p, n-q)}$  of degree  $(n-p, n-q)$  in  $\Omega$  exists and  $\mathcal{H}_{(n-p, n-q)}(\Omega) = \{0\}$ . From the Hodge decomposition, we have for every  $f \in L^2_{(p, q)}(\Omega)$ ,

$$\star f = \bar{\partial}\bar{\partial}^* N_{(n-p, n-q)} \star f + \bar{\partial}^* \bar{\partial} N_{(n-p, n-q)} \star f.$$

We define

$$u = -\star \bar{\partial} N_{(n-p, n-q)} \star f, \quad (1.7)$$

then  $u \in L^2_{(n-p, q-1)}(\Omega)$  and  $\star u \in \text{Dom}(\bar{\partial}^*)$ .

Extending  $u$  to  $\mathcal{X}$  by defining  $u = 0$  in  $\mathcal{X} \setminus \Omega$ , we claim that  $\bar{\partial}u = f$  in the distribution sense in  $\mathcal{X}$ . First we prove that  $\bar{\partial}u = f$  in the distribution sense in  $\Omega$ .

By (1.7) we get

$$\begin{aligned} \bar{\partial}u &= -\bar{\partial} \star \bar{\partial} N_{(n-p, n-q)} \star f \\ &= (-1)^{p+q+1} \star \star \bar{\partial} \star \bar{\partial} N_{(n-p, n-q)} \star f \\ &= (-1)^{p+q} \star \vartheta \bar{\partial} N_{(n-p, n-q)} \star f \\ &= (-1)^{p+q} \star \bar{\partial}^* \bar{\partial} N_{(n-p, n-q)} \star f. \end{aligned} \quad (1.8)$$

It follows from Lemma 1.3 that  $\star f$  is in  $\text{Dom}(\bar{\partial}^*)$  and

$$\bar{\partial}^*(\star f) = 0. \quad (1.9)$$

By our assumption that  $N_{(n-p, n-q-1)}$  exists, we have

$$\bar{\partial}^* N_{(n-p, n-q)} \star f = N_{(n-p, n-q-1)} \bar{\partial}^*(\star f) = 0. \quad (1.10)$$

Combining (1.8) and (1.10) and the assumption  $\mathcal{H}_{(n-p, n-q)}(\Omega) = \{0\}$ , we conclude that

$$\begin{aligned} \bar{\partial}u &= (-1)^{p+q} \star \bar{\partial}^* \bar{\partial} N_{(n-p, n-q)} \star f \\ &= (-1)^{p+q} \star (\bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*) N_{(n-p, n-q)} \star f \\ &= (-1)^{p+q} \star \star f \\ &= f \end{aligned}$$

in the distribution sense in  $\Omega$ . Since  $\star u \in \text{Dom}(\bar{\partial}^*)$ , repeating the same arguments as in (1.6), we have proved  $\bar{\partial}u = f$  in the distribution sense in  $\mathcal{X}$ . Theorem 1.4 is proved.  $\square$

We note that in the proof of Lemma 1.3 and Theorem 1.4, the Lipschitz boundary condition on  $\Omega$  is used to show that the  $C_{(n-p, n-q-1)}^\infty(\bar{\Omega})$  space is dense in  $\text{Dom}(\bar{\partial})$  in the graph norm.

Let  $\Omega \subset\subset \mathbb{C}P^n$  be a pseudoconvex domain with  $C^2$ -smooth boundary  $b\Omega$  and let  $\delta(x) = d(x, b\Omega)$  be the distance function from  $x \in \Omega$  to  $b\Omega$ . We call  $t_0 = t_0(\Omega)$  the order of plurisubharmonicity for the distance function  $\delta$  if

$$t_0(\Omega) = \sup\{0 < \epsilon \leq 1 \mid i\partial\bar{\partial}(-\delta^\epsilon) \geq 0 \text{ on } \Omega\}. \quad (1.11)$$

In  $\mathbb{C}P^n$  with the standard Fubini-Study metric, Ohsawa-Sibony [OS] showed that there exists  $0 < t_0(\Omega) \leq 1$  for any pseudoconvex domain  $\Omega \subset \mathbb{C}P^n$  with  $C^2$ -smooth boundary (see Diederich-Fornaess [DF] for domains in  $\mathbb{C}^n$ ). We recall the following results (see Theorem 2 in [CSW]).

**Theorem 1.5.** *Let  $\Omega$  be a pseudoconvex domain with  $C^2$ -smooth boundary in  $\mathbb{C}P^n$  and let  $t_0$  be the order of plurisubharmonicity for the distance function  $\delta$ . Then the  $\bar{\partial}$ -Neumann operator  $N_{(p,q)}$  exists on  $L_{(p,q)}^2(\Omega)$  where  $0 \leq p, q \leq n$  and the harmonic forms  $\mathcal{H}_{(p,q)}(\Omega) = \{0\}$  if  $1 \leq q \leq n$ . Furthermore,  $N, \bar{\partial}N, \bar{\partial}^*N$  and the Bergman projection  $P$  are exact regular on  $W_{(p,q)}^s(\Omega)$  for  $0 \leq s < \frac{1}{2}t_0$  with respect to the  $W^s(\Omega)$ -Sobolev norms.*

A direct consequence of Theorems 1.2, 1.4 and 1.5 for the case of  $\mathcal{X} = \mathbb{C}P^n$  is the corollary below, which was already obtained in Propositions 4.1 and 4.2 in [CSW].

**Corollary 1.6 ( $L^2$  Cauchy problem for  $\bar{\partial}$  in  $\mathbb{C}P^n$ ).** *Let  $\Omega \subset\subset \mathbb{C}P^n$  be a pseudoconvex domain with  $C^2$  boundary and let  $0 \leq p \leq n$  and  $1 \leq q \leq n$ . For every  $f \in L_{(p,q)}^2(\mathbb{C}P^n)$  supported in  $\bar{\Omega}$ , we assume that  $\bar{\partial}f = 0$  in the distribution sense in  $\mathbb{C}P^n$  if  $1 \leq q \leq n-1$  and  $f$  satisfies (1.2) if  $q = n$ . Then one can find  $u \in L_{(p,q-1)}^2(\mathbb{C}P^n)$  such that  $\bar{\partial}u = f$  in the distribution sense in  $\mathbb{C}P^n$  with  $u$  supported in  $\bar{\Omega}$ .*

*Furthermore, if  $f \in W_{(p,q)}^s(\Omega)$  with  $0 \leq s < \frac{1}{2}t_0$ , then we can choose  $u \in W_{(p,q)}^s(\Omega)$ .*

In the next section, we will show that when the domain is pseudoconvex with  $C^{1,1}$  boundary, then Theorem 1.5 and Corollary 1.6 hold.

## 2. Bounded plurisubharmonic functions for pseudo-convex domains with $C^{1,1}$ boundary

In this section we will recall some results for pseudoconvex domains in  $\mathbb{C}P^n$ . We will also give an alternative proof of the existence of bounded plurisubharmonic functions for domains with  $C^{1,1}$  boundary (see [OS]). Such functions can be used to prove the existence of the  $L^2$   $\bar{\partial}$ -Neumann operators.

**Lemma 2.1.** *Let  $\Omega$  be a Lipschitz pseudoconvex domain with Levi-flat boundary  $M$  in  $\mathbb{C}P^n$ ,  $n \geq 2$ . Then  $M$  is locally foliated by complex hypersurfaces. Moreover, for each  $Q \in M$ , there exist a neighborhood  $U$  of  $Q$  and local unitary frame  $\{\tilde{e}_1, \dots, \tilde{e}_{n-1}, \tilde{e}_n\}$  on  $U$  such that (1) for  $z \in M \cap U$ , the vector fields  $\{\tilde{e}_1, \dots, \tilde{e}_{n-1}\}|_z$  are tangent to the leaves of the foliation of  $M \cap U$ ; and (2) The covariant derivative  $\nabla_\xi \tilde{e}_j$  is a bounded function for  $j = 1, \dots, n-1$  and any unit vector  $\xi \in T_z(\mathbb{C}P^n)$  with  $z \in U$ .*

*Proof.* Since  $M$  is Levi-flat, it is locally foliated by complex manifolds of dimension  $n-1$  and the foliation is Lipschitz in the transversal direction. For any point  $Q \in M$ , we can parametrize a neighborhood  $V \subset M$  of  $Q$  as follows. Let  $\{z', g(z', t)\}$  denote the leaf  $\Sigma_t$  where  $g(z', t)$  is holomorphic in  $z' = (z_1, \dots, z_{n-1}) \in \mathbb{B}_\epsilon \subset \mathbb{C}^{n-1}$  and Lipschitz in  $t$  for  $0 \leq |t| < \mu$ . We can parametrize  $M$  locally as a graph of the function  $g$ , by setting

$$\Psi(z', t) = (z', g(z', t)),$$

where  $z' \in \mathbb{C}$ ,  $0 \leq |t| < \mu$ . Clearly,  $\Psi : \mathbb{B}_\epsilon \times (-\mu, \mu) \rightarrow M$  is a local coordinate map of  $M$  and  $\Psi$  is Lipschitz in  $t$  and  $C^\infty$  (holomorphic) in  $z'$ .

Let  $z' = (z_1, \dots, z_{n-1})$  and extend  $\Psi$  to a map  $\tilde{\Psi} : \mathbb{B}_\epsilon \times (-\mu, \mu) \times (-\mu, \mu) \rightarrow \mathbb{C}P^n$  by setting  $\tilde{\Psi}(z', t + is) = (z', g(z', t) + s\vec{v}_0)$ , where  $(t, s) \in (-\mu, \mu) \times (-\mu, \mu)$  and  $\vec{v}_0$  is a constant vector transversal to  $\frac{\partial g(z', t)}{\partial t}$  for all  $(z', t) \in \mathbb{B}_\epsilon \times (-\mu, \mu)$ . We now choose  $\tilde{v}_j = \frac{\partial \tilde{\Psi}}{\partial z_j}$  for  $j = 1, \dots, n-1$ . Applying the Gram-Schmidt process to the frame  $\{\tilde{v}_1, \dots, \tilde{v}_{n-1}\}$ , we obtain a unitary frame  $\{\tilde{e}_1, \dots, \tilde{e}_{n-1}\}$  with Lipschitz coefficients. Thus (2) is satisfied as desired.  $\square$

We recall the following theorem by [Ta] (see also [CS]).

**Theorem 2.2.** *Let  $\Omega \subset\subset \mathbb{C}P^n$  be a pseudoconvex domain. Then the distance function  $\delta$  satisfies*

$$i\partial\bar{\partial}(-\log \delta) \geq \omega \quad (2.1)$$

*as currents where  $\omega$  is the Kähler form of the Fubini-Study metric on  $\mathbb{C}P^n$ .*

In  $\mathbb{C}P^n$  with the standard Fubini-Study metric, Ohsawa-Sibony [OS] showed that there exists a bounded plurisubharmonic functions for pseudoconvex domains with  $C^2$  boundary. We give a proof below for pseudoconvex domains with  $C^{1,1}$  boundary.

**Proposition 2.3.** *Let  $\Omega \subset\subset \mathbb{C}P^n$  be a pseudoconvex domain with  $C^{1,1}$  boundary  $b\Omega$ . Then there exists a distance function  $\delta$  in  $C^{1,1}(\bar{\Omega})$  which satisfies (2.1) almost everywhere. Furthermore, there exists  $t_0 = t_0(\Omega)$  with  $0 < t_0 \leq 1$  such that*

$$i\partial\bar{\partial}(-\delta^{t_0}) \geq 0. \quad (2.2)$$

*Proof.* Let  $\delta$  be the distance function from  $z \in \Omega$  to  $b\Omega$ . Since the boundary is of class  $C^{1,1}$ , we have that there exists a neighborhood  $U$  of  $b\Omega$  such that  $\delta$  is in



$C^{1,1}(\bar{\Omega} \cap U)$ . Using [Ta], we have

$$i\partial\bar{\partial}(-\log \delta) = i\frac{\partial\bar{\partial}(-\delta)}{\delta} + \frac{i\partial\delta \wedge \bar{\partial}\delta}{\delta^2} \geq \omega \quad (2.3)$$

near the boundary almost everywhere.

To prove (2.2), observe that inequality (2.2) is equivalent to

$$i\frac{\partial\bar{\partial}(-\delta)}{\delta} + (1-t_0)\frac{i\partial\delta \wedge \bar{\partial}\delta}{\delta^2} \geq 0. \quad (2.4)$$

Compare (2.4) with (2.3), we see that (2.2) is equivalent to

$$i\partial\bar{\partial}(-\log \delta) \geq t_0 \frac{i\partial\delta \wedge \bar{\partial}\delta}{\delta^2}. \quad (2.5)$$

Near a boundary point, we choose a special orthonormal basis  $w_1, \dots, w_n$  for  $(1,0)$ -forms such that  $w_n = \sqrt{2}\partial(-\delta)$ . Let  $L_1, \dots, L_n$  be its dual and let  $a$  be any  $(1,0)$ -vector. We decompose  $a = a_\tau + a_\nu$  where  $a_\nu = \langle a, L_n \rangle$  is the complex normal component and  $a_\tau$  is the complex tangential component. We have

$$\begin{aligned} & \langle \partial\bar{\partial}(-\log \delta), a \wedge \bar{a} \rangle \\ &= \langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\tau \rangle + 2\Re \langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\nu \rangle \\ &+ \langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\nu \wedge \bar{a}_\nu \rangle + \frac{|a_\nu|^2}{\delta^2}. \end{aligned} \quad (2.6)$$

From (2.1) and (2.3), we have

$$\langle \partial\bar{\partial}(-\log \delta), a_\tau \wedge \bar{a}_\tau \rangle \geq \langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\tau \rangle \geq |a_\tau|^2.$$

Thus from (2.6),

$$\begin{aligned} \langle \partial\bar{\partial}(-\log \delta), a \wedge \bar{a} \rangle &\geq |a_\tau|^2 + \frac{|a_\nu|^2}{\delta^2} - 2|\langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\nu \rangle| \\ &- |\langle \frac{\partial\bar{\partial}(-\delta)}{\delta}, a_\nu \wedge \bar{a}_\nu \rangle|. \end{aligned} \quad (2.7)$$

Using the assumption that  $b\Omega$  is  $C^{1,1}$ , we have

$$|\partial\bar{\partial}\rho| \leq C \quad (2.8)$$

Also for any  $\epsilon > 0$ , there exists a small neighborhood  $U$  of  $b\Omega$  such that

$$|\partial\bar{\partial}\delta| \leq \frac{\epsilon}{\delta}. \quad (2.9)$$

Thus for any  $\epsilon > 0$ , we have from (2.8),

$$|\langle \frac{\partial \bar{\partial}(-\delta)}{\delta}, a_\tau \wedge \bar{a}_\nu \rangle| \leq C \left( \frac{1}{\epsilon} |a_\tau|^2 + \epsilon \frac{|a_\nu|^2}{\delta^2} \right), \quad (2.10)$$

and from (2.9),

$$|\langle \frac{\partial \bar{\partial}(-\delta)}{\delta}, a_\nu \wedge \bar{a}_\nu \rangle| \leq \frac{\epsilon}{\delta^2} |a_\nu|^2 \quad (2.11)$$

on a sufficiently small neighborhood  $U$  of the boundary.

Substituting (2.9)-(2.10) into (2.7) and choosing  $\epsilon$  sufficiently small, we have

$$\langle \partial \bar{\partial}(-\log \delta), a \wedge \bar{a} \rangle \geq \frac{1}{2} \frac{|a_\nu|^2}{\delta^2} - K |a_\tau|^2 \quad (2.12)$$

for some large constant  $K$  depending on  $\epsilon$ . Multiplying (2.1) by  $K$  and adding it to (2.12), we have

$$(K+1) \langle \partial \bar{\partial}(-\log \delta), a \wedge \bar{a} \rangle \geq \frac{1}{2} \frac{|a_\nu|^2}{\delta^2}.$$

This proves (2.5) with  $t_0 = \frac{1}{2(K+1)}$  near the boundary, or equivalently, (2.2) is proved near the boundary. Since  $\Omega$  is Stein, on any relatively compact submanifold  $\Omega' \subset\subset \Omega$ , there exists a bounded strictly plurisubharmonic function on  $\overline{\Omega}'$ . By standard arguments one can extend  $\delta$  so that  $\delta$  is the distance function near the boundary and  $\delta$  satisfies (2.1) and (2.2) in  $\Omega$ .  $\square$

**Remark:** Diederich-Fornaess [DF] show that if  $\Omega$  is a pseudoconvex domain in  $\mathbb{C}^n$  with  $C^2$  boundary, let  $\tilde{\delta} = \delta e^{-K|z|^2}$  with large  $K > 0$ , then (2.1) holds with  $\delta$  substituted by  $\tilde{\delta}$ . The proof of Proposition 2.3 is a modified proof of the Diederich-Fornaess [DF] and Ohsawa-Sibony [OS] results. We also remark that bounded plurisubharmonic exhaustion functions exist for pseudoconvex domains in  $\mathbb{C}^n$  with  $C^1$  (see Kerzman-Rosay [KeR]) or even Lipschitz boundary (see Demailly [De]), but it is not known if such functions exist for  $C^1$  or Lipschitz pseudoconvex domains in  $\mathbb{C}P^n$ .

**Proposition 2.4.** *Let  $\Omega$  be a pseudo-convex domain with  $C^{1,1}$ -smooth boundary in  $\mathbb{C}P^n$ ,  $n \geq 2$ . Then the  $\bar{\partial}$ -Neumann operator  $N_{(p,q)}$  exists on  $L^2_{(p,q)}(\Omega)$  where  $0 \leq p, q \leq n$  and the harmonic forms  $\mathcal{H}_{(p,q)}(\Omega) = \{0\}$  if  $1 \leq q \leq n$ . Furthermore, there exist  $t_0 > 0$  such that  $N, \bar{\partial}N, \bar{\partial}^*N$  and the Bergman projection  $P$  are exact regular on  $W^s_{(p,q)}(\Omega)$  for  $0 < s < \frac{1}{2}t_0$  with respect to the  $W^s(\Omega)$ -Sobolev norms.*

*Proof.* Let  $\Omega$ ,  $\delta$  and  $t_0$  be the same as in Proposition 2.4. The proposition follows exactly the same as the proof of Theorem 2 in [CSW].

From Proposition 2.4, the results of Theorem 1.5 and Corollary 1.6 hold also for  $C^{1,1}$  pseudoconvex domains. Then we can use the same arguments as in Section

5 in [CSW] to show the nonexistence of  $C^{1,1}$  Levi-flat hypersurfaces in  $\mathbb{C}P^n$  when  $n \geq 3$ . But it is not known if Proposition 2.3 holds for Lipschitz domains. In the next section, we will use the weighted  $\bar{\partial}$ -Neumann operators to study the  $\bar{\partial}$ -Cauchy problem on Lipschitz domains.

### 3. The $\bar{\partial}$ -Cauchy problem with weights on Lipschitz pseudoconvex domains in $\mathbb{C}P^n$

Let  $\Omega$  be a pseudoconvex domain with Lipschitz boundary in  $\mathbb{C}P^n$ ,  $n \geq 2$ . We study the  $\bar{\partial}$ -Cauchy problem with weights and the  $\bar{\partial}$ -closed extension of forms from pseudoconcave domains.

For  $t > 0$ , let  $L^2(e^{-\phi_t}, \Omega) = L^2(\delta^t, \Omega) = L^2(\delta^t)$  be the weighted  $L^2$  space with respect to the weight function  $\phi_t = -t \log \delta$ . The norm in  $L^2(\delta^t)$  is denoted by  $\|\cdot\|_{(t)}$ . Let  $\bar{\partial}$  and  $\bar{\partial}_t^*$  be the closure of  $\bar{\partial}$  and its  $L^2$  adjoint with respect to the weighted  $L^2(\delta^t)$  space.

**Proposition 3.1.** *Let  $\Omega \subset\subset \mathbb{C}P^n$  be a pseudoconvex domain. For any  $t > 0$  and  $(p, q)$ -form  $f \in L^2(\delta^t)$ , where  $0 \leq p \leq n$  and  $1 \leq q \leq n$ , such that  $\bar{\partial}f = 0$  in  $\Omega$ , there exists  $u \in L^2_{(p, q-1)}(\delta^t)$  satisfying  $\bar{\partial}u = f$  and*

$$\|u\|_{(t)}^2 \leq \frac{1}{t} \|f\|_{(t)}^2. \quad (3.1)$$

Furthermore, the weighted  $\bar{\partial}$ -Neumann operator  $N_t$  exists for all  $t > 0$ .

*Proof.* We first assume that  $\Omega$  is  $C^2$ . By [Ta], we have that  $\phi = -\log \delta$  is strictly plurisubharmonic and  $i\partial\bar{\partial}\phi \geq \omega$ , where  $\omega$  is the Kähler form of  $\mathbb{C}P^n$  with the Fubini-Study metric. Using Hörmander's weighted  $L^2$  estimates for the  $\bar{\partial}$ -Neumann problem (see e.g. Proposition A.4 in [CSW]), we have the following formula: for any  $(p, q)$ -form  $g \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}_t^*)$ ,

$$\|\bar{\partial}g\|_{(t)}^2 + \|\bar{\partial}_t^*g\|_{(t)}^2 \geq t((i\partial\bar{\partial}\phi)g, \bar{g})_{(t)}. \quad (3.2)$$

Thus, we have

$$\|\bar{\partial}g\|_{(t)}^2 + \|\bar{\partial}_t^*g\|_{(t)}^2 \geq t\|g\|_{(t)}^2. \quad (3.3)$$

For any  $f \in L^2(\delta^t)$ , there exists  $u \in L^2(\delta^t)$  satisfying  $\bar{\partial}u = f$  and (3.1). This proves the proposition when  $\Omega$  is  $C^2$ . The general case follows from approximating the domain  $\Omega$  from inside by smooth pseudoconvex domains.  $\square$

From (3.1), we have that the weighted  $\bar{\partial}$ -Neumann operator  $N_t$  exists for each  $t > 0$  (see the proof of Theorem 4.4.1 in [CS]).  $\square$

We remark that there is no smoothness assumption on the boundary  $b\Omega$  in Proposition 3.1. We will use the weighted  $\bar{\partial}$ -Neumann operator  $N_t$  to study the  $\bar{\partial}$ -Cauchy problem.

**Proposition 3.2.** *Let  $\Omega \subset \subset \mathbb{C}P^n$  be a pseudoconvex domain with Lipschitz boundary,  $n \geq 3$ . Suppose that  $f \in L^2_{(p,q)}(\delta^{-t}, \Omega)$  for some  $t > 0$ , where  $0 \leq p \leq n$  and  $1 \leq q < n$ . Assuming that  $\bar{\partial}f = 0$  in  $\mathbb{C}P^n$  with  $f = 0$  outside  $\Omega$ , then there exists  $u_t \in L^2_{(p,q-1)}(\delta^{-t}, \Omega)$  with  $u_t = 0$  outside  $\Omega$  satisfying  $\bar{\partial}u_t = f$  in the distribution sense in  $\mathbb{C}P^n$ .*

*Proof.* From Proposition 3.1, the weighted  $\bar{\partial}$ -Neumann operators  $N_t$  exists for forms in  $L^2_{(n-p,n-q)}(\delta^t, \Omega)$ . Let  $\star_{(t)}$  denote the Hodge-star operator with respect to the weighted norm  $L^2(\delta^t, \Omega)$ . Then

$$\star_{(t)} = \delta^t \star = \star \delta^t$$

where  $\star$  is the Hodge star operator with the unweighted  $L^2$  norm. Since  $f \in L^2_{(p,q)}(\delta^{-t}, \Omega)$ , we have that  $\star_{(-t)}f \in L^2_{(p,q)}(\delta^t, \Omega)$ . Let  $u_t$  be defined by

$$u_t = -\star_{(t)} \bar{\partial}N_t \star_{(-t)}f. \quad (3.4)$$

Then  $u_t \in L^2_{(p,q-1)}(\delta^{-t}, \Omega)$ , since  $\bar{\partial}N_t \star_{(-t)}f$  is in  $\text{Dom}(\bar{\partial}_t^*) \subset L^2_{(n-p,n-q+1)}(\delta^t, \Omega)$ . Since  $\bar{\partial}_t^* = \delta^{-t} \vartheta \delta^t = -\star_{(-t)} \bar{\partial} \star_{(t)}$ , using the same proof as in Lemma 1.3, we have  $\star_{(-t)}f \in \text{Dom}(\bar{\partial}_t^*)$  and  $\bar{\partial}_t^* \star_{(-t)}f = 0$  in  $\Omega$ . This gives

$$\bar{\partial}_t^* N_t \star_{(-t)}f = N_t \bar{\partial}_t^* \star_{(-t)}f = 0. \quad (3.5)$$

From (3.5), we have

$$\begin{aligned} \bar{\partial}u_t &= -\bar{\partial}(\star_{(t)} \bar{\partial}N_t \star_{(-t)}f) \\ &= (-1)^{p+q} \star_{(t)} \bar{\partial}_t^* \bar{\partial}N_t \star_{(-t)}f \\ &= (-1)^{p+q} \star_{(t)} \bar{\partial}_t^* \bar{\partial}N_t \star_{(-t)}f + (-1)^{p+q} \star_{(t)} \bar{\partial} \bar{\partial}_t^* N_t \star_{(-t)}f \\ &= (-1)^{p+q} \star_{(t)} \star_{(-t)}f \\ &= f \quad \text{in } \Omega. \end{aligned} \quad (3.6)$$

First notice that  $\star_{(-t)}(-1)^{p+q} \bar{\partial}N_t \star_{(-t)}f = \bar{\partial}N_t \star_{(-t)}f \in \text{Dom}(\bar{\partial}_t^*)$ . We also have  $\bar{\partial}_t^* \star_{(-t)}u = (-1)^{p+q} \star_{(-t)}f$  in  $\Omega$ . Extending  $u_t$  to be zero outside  $\Omega$ , one can show that  $\bar{\partial}u_t = f$  in  $\mathbb{C}P^n$ . The proof is similar to the proof of Theorem 1.2. In fact, for any  $\psi \in C^\infty_{(p,q)}(\mathbb{C}P^n)$ ,

$$\begin{aligned} (u, \vartheta\psi)_{\mathbb{C}P^n} &= (\star\vartheta\psi, \star_{(-t)}u)_{(t)\Omega} \\ &= (-1)^{p+q} (\bar{\partial}\star\psi, \star_{(-t)}u)_{(t)\Omega} \\ &= (-1)^{p+q} (\star\psi, \bar{\partial}_t^*(\star_{(-t)}u))_{(t)\Omega} \\ &= (\star\psi, \star_{(-t)}f)_{(t)\Omega} = (\star\psi, \star f)_{\Omega} \\ &= (f, \psi)_{\mathbb{C}P^n}, \end{aligned} \quad (3.7)$$

where the third equality holds since  $\star_{(-t)}u \in \text{Dom}(\bar{\partial}_t^*)$ . Thus  $\bar{\partial}u = f$  in the distribution sense in  $\mathbb{C}P^n$ .  $\square$

**Theorem 3.3.** *Let  $\Omega \subset \subset \mathbb{C}P^n$  be a pseudoconvex domain with Lipschitz boundary and let  $\Omega^+ = \mathbb{C}P^n \setminus \bar{\Omega}$ . For any  $f \in W_{(p,q)}^{1+\epsilon}(\Omega^+)$ , where  $0 \leq p \leq n$ ,  $0 \leq q < n-1$  and  $0 < \epsilon < \frac{1}{2}$ , such that  $\bar{\partial}f = 0$  in  $\Omega^+$ , there exists  $F \in W_{(p,q)}^\epsilon(\mathbb{C}P^n)$  with  $F|_{\Omega^+} = f$  and  $\bar{\partial}F = 0$  in  $\mathbb{C}P^n$  in the distribution sense.*

*Proof.* Since  $\Omega$  has Lipschitz boundary, there exists a bounded extension operator from  $W^s(\Omega^+)$  to  $W^s(\mathbb{C}P^n)$  for all  $s \geq 0$  (see e.g. [Gr] or [St]). Let  $\tilde{f} \in W_{(p,q)}^{1+\epsilon}(\mathbb{C}P^n)$  be the extension of  $f$  so that  $\tilde{f}|_{\Omega^+} = f$  with  $\|\tilde{f}\|_{W^{1+\epsilon}(\mathbb{C}P^n)} \leq C\|f\|_{W^{1+\epsilon}(\Omega^+)}$ . Furthermore, we can choose an extension such that  $\bar{\partial}\tilde{f} \in W^\epsilon(\Omega) \cap L^2(\delta^{-2\epsilon}, \Omega)$ .

We define  $T\tilde{f}$  by  $T\tilde{f} = -\star_{(2\epsilon)}\bar{\partial}N_{2\epsilon}(\star_{(-2\epsilon)}\bar{\partial}\tilde{f})$  in  $\Omega$ . From Proposition 3.2, we have that  $T\tilde{f} \in L^2(\delta^{-2\epsilon}, \Omega)$ . But for a Lipschitz domain, we have that  $T\tilde{f} \in L^2(\delta^{-2\epsilon}, \Omega)$  is comparable to  $W^\epsilon(\Omega)$  when  $0 < \epsilon < \frac{1}{2}$ . This gives that  $T\tilde{f} \in W^\epsilon(\Omega)$  and  $T\tilde{f}$  satisfies  $\bar{\partial}T\tilde{f} = \bar{\partial}\tilde{f}$  in  $\mathbb{C}P^n$  in the distribution sense if we extend  $T\tilde{f}$  to be zero outside  $\Omega$ .

Since  $0 < \epsilon < \frac{1}{2}$ , the extension by 0 outside  $\Omega$  is a continuous operator from  $W^\epsilon(\Omega)$  to  $W^s(\mathbb{C}P^n)$  (see e.g. [LM] or [Gr]). Thus we have  $T\tilde{f} \in W^\epsilon(\mathbb{C}P^n)$ .

Define

$$F = \begin{cases} f, & x \in \bar{\Omega}^+, \\ \tilde{f} - T\tilde{f}, & x \in \Omega. \end{cases}$$

Then  $F \in W_{(p,q)}^\epsilon(\mathbb{C}P^n)$  and  $F$  is a  $\bar{\partial}$ -closed extension of  $f$ .  $\square$

**Corollary 3.4.** *Let  $\Omega^+$  be a pseudoconcave domain in  $\mathbb{C}P^n$  with Lipschitz boundary, where  $n \geq 2$ . Then  $W_{(p,0)}^{1+\epsilon}(\Omega^+) \cap \text{Ker}(\bar{\partial}) = \{0\}$  for every  $1 \leq p \leq n$  and  $W_{(0,0)}^{1+\epsilon}(\Omega^+) \cap \text{Ker}(\bar{\partial}) = \mathbb{C}$ .*

*Proof.* Using Theorem 3.3 for  $q = 0$ , we have that any holomorphic  $(p, 0)$ -form on  $\Omega^+$  extends to be a holomorphic  $(p, 0)$  in  $\mathbb{C}P^n$ , which are zero (when  $p > 0$ ) or constants (when  $p = 0$ ).

**Corollary 3.5.** *Let  $\Omega^+$  be a pseudoconcave domain in  $\mathbb{C}P^n$  with Lipschitz boundary, where  $n \geq 3$ . For any  $f \in W_{(p,q)}^{1+\epsilon}(\Omega^+)$ , where  $0 \leq p \leq n$ ,  $1 \leq q < n-1$ ,  $p \neq q$  and  $0 < \epsilon < \frac{1}{2}$ , such that  $\bar{\partial}f = 0$  in  $\Omega^+$ , there exists  $u \in W_{(p,q-1)}^{1+\epsilon}(\Omega^+)$  with  $\bar{\partial}u = f$  in  $\Omega^+$ .*

#### 4. Nonexistence of Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ when $n \geq 3$

In this section we study  $\bar{\partial}_b$ -exactness of  $(0, 1)$ -form  $f$  on a Lipschitz Levi-flat hypersurface  $M \subset \mathbb{C}P^n$  and prove the main theorem. It is a refinement of arguments used in [Si1] and [CSW].

We recall the definition of the Chern connection form for the complex line bundle generated by the complex normal of  $M$ . Let  $\mathbb{C}P^n \setminus M = \Omega^+ \cup \Omega^-$ . Let  $\rho$  be the

signed distance function of  $M$

$$\rho(z) = \begin{cases} -d(z, M), & \text{if } z \in \Omega^-, \\ d(z, M), & \text{if } z \in \overline{\Omega^+}. \end{cases} \quad (4.1)$$

If  $J$  is the complex structure of  $\mathbb{C}P^n$  and  $\nabla$  is the covariant derivative of  $\mathbb{C}P^n$  with respect to the Fubini-Study metric, the connection form of the complex normal line bundle  $\nabla\rho \otimes \mathbb{C}$  on  $M$  is given by

$$\beta(X) = \langle \nabla_X(\nabla\rho), J\nabla\rho \rangle = -\langle \nabla_X(J\nabla\rho), \nabla\rho \rangle, \quad (4.2)$$

where  $X$  is a tangent vector on  $M$  (see (5.3) and (A.7) in [CSW]).

For a general hypersurface, we need  $C^2$  smoothness to define the curvature form and the connection form. In this case, the curvature form  $\tilde{\Theta}^N$  associated with the complex line bundle for  $M$  is a well-defined 2-form with  $C^0$  coefficients in  $U$  and is  $d$ -exact. Following the Chern formula (see Proposition A.1 in [CSW]), we have that  $\tilde{\Theta}^N = \sqrt{-1}d\beta$  on a tubular neighborhood  $U(M)$  of  $M$  in  $\mathbb{C}P^n$ . Let  $\beta_b$  be the projection of  $\beta$  to  $M$  defined by

$$\beta_b = \beta|_{T^{(1,0)}(M) \oplus T^{(0,1)}(M)}.$$

Write  $\beta_b = \beta_b^{1,0} + \beta_b^{0,1}$  where  $\beta_b^{1,0}$  and  $\beta_b^{0,1}$  are the  $(1,0)$  and  $(0,1)$  components of  $\beta_b$ . When the hypersurface  $M$  is Levi-flat, one can relax the smoothness using Lemma 2.1. We first show that the Chern connection and the curvature can be defined for Lipschitz hypersurfaces.

**Lemma 4.1.** *Let  $M$  be a Lipschitz Levi-flat hypersurface in  $\mathbb{C}P^n$ ,  $n \geq 2$ . Then the curvature form  $\tilde{\Theta}^N$  associated with the complex line bundle for  $M$  is a well-defined 2-form with  $L^\infty$  coefficients in  $M$  and is  $d$ -exact. In fact, we have  $\tilde{\Theta}^N = \sqrt{-1}d\beta$  for some form  $\beta$  on a tubular neighborhood  $U(M)$  of  $M$  in  $\mathbb{C}P^n$ . Furthermore, we can choose  $\beta$  to be  $C^{1-\gamma}$ -smooth for any small  $\gamma > 0$ .*

*Proof.* Let  $Q$  be a point on  $M$  and  $\Sigma_Q$  be the holomorphic leaf of  $M$  passing through  $Q$  with  $\dim_{\mathbb{C}}[\Sigma_Q] = n - 1$ . There is a holomorphic coordinate system  $(z_1, z_2, \dots, z_n)$  of  $\mathbb{C}P^n$  near  $Q$ , such that  $(z_1, \dots, z_{n-1})$  is a local coordinate system of  $\Sigma_Q$  near  $Q$ . Applying the Gram-Schmidt process to the local holomorphic frame  $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$  near  $Q$ , we obtain a special unitary basis  $\tilde{e}_1, \dots, \tilde{e}_n$  such that  $\tilde{e}_l \in T^{1,0}(M)$  for  $l = 1, \dots, n - 1$  and  $\tilde{e}_n|_P$  is orthogonal to  $T_P^{(1,0)}(\Sigma_Q)$  for all  $P \in \Sigma_Q$ , with respect to the Fubini-Study metric. If  $M$  is  $C^1$ , then  $\tilde{e}_n = \lambda(\partial\rho)_\#$  for some  $\lambda$  with  $|\lambda| = \sqrt{2}$ . Notice that  $\lambda$  is not necessarily a real valued function in  $P \in \Sigma_Q$ . Let  $\tilde{\theta}_{n,\bar{l}}$  be the connection 1-forms with respect to a unitary basis  $\tilde{e}_1, \dots, \tilde{e}_n$  with  $\tilde{e}_j \in T^{1,0}(M)$  for  $j = 1, \dots, n - 1$ . It is well-known that the curvature form  $\tilde{\Theta}^N$  of the quotient line bundle  $T^{(1,0)}(\mathbb{C}P^n)/T^{(1,0)}(\Sigma_Q)$  is independent of the choice of

local frame  $\{\tilde{e}_1, \dots, \tilde{e}_n\}$ . Furthermore, its curvature form  $\tilde{\Theta}^N$  is a closed form, by the Chern-Weil theory. We remark that the Chern classes are well defined for any continuous complex vector bundle (see [Mi]).

To see that  $\tilde{\Theta}^N$  has  $L^\infty$  coefficients, we use the generalized Gauss-Codazzi equations (the Cartan-Chern structure formula, see (A.14)-(A.17) in [CSW] and the notation therein). Using Lemma 2.1, each  $\tilde{\theta}_{n,\bar{l}}$  has bounded measurable coefficients. Let  $\tilde{\Theta}$  denote the curvature tensor for  $\mathbb{C}P^n$  which is an  $n \times n$  matrix and  $\Theta_{n,\bar{n}}$  be its  $(n, \bar{n})$  component. We have

$$\tilde{\Theta}^N = \tilde{\Theta}_{n,\bar{n}} - \sum_{l=1}^{n-1} \tilde{\theta}_{n,\bar{l}} \wedge \tilde{\theta}_{l,\bar{n}}, \quad (4.5)$$

where  $\theta_{l,\bar{n}}$  is given by

$$\tilde{\theta}_{j,\bar{n}}(\xi) = -\langle \nabla_\xi(\tilde{e}_n), \tilde{e}_j \rangle = \langle \nabla_\xi(\tilde{e}_j), \tilde{e}_n \rangle, \quad \xi \in T(\mathbb{C}P^n),$$

and  $\theta_{j,\bar{n}} = -\bar{\theta}_{n,\bar{j}}$  (see (A.17)-(A.18) of [CSW]). This gives that  $\tilde{\Theta}^N$  has bounded coefficients on  $M$ .

Because  $M$  has real codimension 1 in  $\mathbb{C}P^n$  and  $M$  is locally the graph of some Lipschitz function, using a partition of unity,  $M$  admits a nowhere vanishing continuous global cross-section  $\{\zeta\}$  in the quotient line bundle  $\mathcal{L} = T^{(1,0)}(\mathbb{C}P^n)/T^{(1,0)}(M)$ . The quotient line bundle  $\mathcal{L}$  is topologically trivial on  $M$ , just as in the smooth case (see [Si1]).

This line bundle  $\mathcal{L}$  can be extended trivially to a small neighborhood  $U(M)$  of  $M$ . Let  $\widehat{M}_s = \rho^{-1}(s)$ . Then  $\widehat{M}_s$  gives rise to a family of Lipschitz hypersurfaces for all  $|s| < \epsilon$ ,  $\epsilon > 0$  small, with  $\widehat{M}_0 = M$ . Using the mollifier smoothing technique (cf [Ka]), one can obtain a family of smooth hypersurfaces  $\tilde{M}_s$  such that each  $\tilde{M}_s$  is a smooth real hypersurface when  $s > 0$  and  $\tilde{M}_0 = M$ . Let  $U_{\epsilon_0} = \cup_{|s| < \epsilon_0} \tilde{M}_s$  for some small  $\epsilon_0 > 0$ . Then  $U_{\epsilon_0}$  is an open neighborhood of  $M$ . The complex line bundle  $\tilde{\mathcal{L}}$  on  $U_{\epsilon_0}$  induced by  $T^{(1,0)}(\mathbb{C}P^n)/T^{(1,0)}(\tilde{M}_s)$  is topologically trivial on  $U_{\epsilon_0}$  since  $\{\zeta\}$  is a nowhere vanishing continuous cross section. Also  $\tilde{\mathcal{L}}|_M = \mathcal{L}$ . Thus the Chern curvature form  $\tilde{\Theta}^N$  is  $d$ -exact in  $U_{\epsilon_0}$ . Using (4.5) again, we see that  $\tilde{\Theta}^N$  has  $L^\infty$  coefficients in  $U_{\epsilon_0} = U(M)$ .

Since  $\tilde{\Theta}^N$  is  $d$ -exact on  $U(M)$  and  $L^\infty \subset C^{-\gamma}$  on  $U(M)$  for any  $\gamma > 0$ , we can use the de Rham-Hodge decomposition theorem and interior regularity of the  $d$ -operator on  $U(M)$  to find some  $\beta$ , which is  $C^{1-\gamma}$ -smooth for arbitrarily small  $\gamma > 0$ .  $\square$

**Proposition 4.2.** *Let  $M$  be a compact Lipschitz Levi-flat hypersurface in  $\mathbb{C}P^n$ ,  $n \geq 3$ . Let  $\beta_b^{0,1}$  be the projection of the Chern connection form  $\beta$  to  $T^{0,1}(M)$ , where  $\beta \in C^{1-\gamma}(M)$  is given by Lemma 4.1. Then there exists an  $\epsilon' > 0$  and a function  $u \in C^{\epsilon'}(M)$  such that*

$$\bar{\partial}_b u = \beta_b^{0,1} \quad \text{in } M.$$

*Proof.* Let  $\mathbb{C}P^n \setminus M = \Omega^+ \cup \Omega^-$ . Then  $\Omega^+$  and  $\Omega^-$  are pseudoconvex domains with Lipschitz Levi-flat boundary. From Lemma 4.1,  $\beta_b^{(0,1)}$  has  $C^{1-\gamma}$  coefficients where  $0 < \gamma < 1$  on  $M$ .

Since  $M$  is Lipschitz, using the trace theorem (see [Gr]), we can extend  $\beta_b^{0,1}$  to an  $(0,1)$ -form  $\tilde{\beta}^{0,1}$  on the whole  $\mathbb{C}P^n$  such that  $\tilde{\beta}^{0,1} \in W^{1-\gamma+\frac{1}{2}}(\mathbb{C}P^n)$ . Let  $0 < \varepsilon = \frac{1}{2} - \gamma < \frac{1}{2}$ . Then  $f = \bar{\partial}\tilde{\beta}^{0,1} \in W^s(\mathbb{C}P^n)$ . We set  $f_{\pm} = f|_{\Omega^{\pm}}$ . We may choose our extension such that that  $f_{\pm} \in L^2_{(0,2)}(\delta^{-t}, \Omega)$  for  $t = 2\varepsilon$  since  $\varepsilon < \frac{1}{2}$ . Applying the proof of Proposition 3.2 and Theorem 3.3, we observe that

$$\hat{\beta}_{\pm}^{0,1} = \tilde{\beta}^{0,1} + \star_{(t)} \bar{\partial} N_t \star_{(-t)} f_{\pm}$$

is a  $\bar{\partial}$ -closed extension of  $\beta_b^{0,1}$  to  $\Omega^{\pm}$ . Thus,  $\beta_b^{0,1}$  has a  $\bar{\partial}$ -closed extension  $\hat{\beta}^{0,1}$  on the whole  $\mathbb{C}P^n$ , with  $\hat{\beta}^{0,1} \in W^{\varepsilon}_{(0,1)}(\mathbb{C}P^n)$ . Since the cohomology group  $H^{(0,1)}(\mathbb{C}P^n)$  vanishes. We can find  $u \in W^{1+\varepsilon}(\mathbb{C}P^n)$  with

$$\bar{\partial}u = \hat{\beta}^{0,1}.$$

Using the trace theorem again, we conclude that there is a  $u \in W^{\frac{1}{2}+\varepsilon}(M)$  such that

$$\bar{\partial}_b u = \beta_b^{0,1} \quad \text{on } M. \quad (4.6)$$

Using the local parametrization used in Lemma 2.1 with  $V = \cup_{|t|<\mu} \Sigma_t \subset M$ , the equation  $\bar{\partial}_b$  is equal to  $\bar{\partial}_{z'}$  on each leaf  $\Sigma_t$ , which is elliptic. From Lemma 4.1,  $\beta_b^{0,1}$  is  $C^{\alpha}$  on  $M$ . From (4.6), and the classic Schauder theorem (cf. [GT]) for elliptic equations on  $\Sigma_t$ , we get that  $u$  is  $C^{1,\alpha}$ -smooth on each leaf. Furthermore, we have (see e.g. [ShW]) that there exists a constant  $C$  independent of  $t$  such that

$$\|u\|_{C^{1+\alpha}(\Sigma_t)} \leq C(\|\beta_b^{0,1}\|_{C^{\alpha}(\Sigma_t)} + \|u\|_{L^2(\Sigma_t)}), \quad (4.7)$$

where  $C$  depends on the neighborhood  $V$  of  $Q$  and the parametrization  $\Psi$ , but is independent of  $t$  since (4.6) is uniformly elliptic on  $\Sigma_t \subset V$  independent of  $t$ .

From the Sobolev trace theorem, the function  $u \in W^{\frac{1}{2}+\varepsilon}(M)$  has  $L^2$ -trace on each leaf. Therefore, there exists  $C_2 > 0$  independent of  $t$  such that

$$\|u\|_{L^2(\Sigma_t)} \leq C_2 \|u\|_{W^{\frac{1}{2}+\varepsilon}(M)}. \quad (4.8)$$

Combining (4.7) and (4.8), we get

$$\|u\|_{L^{\infty}(V)} \leq \sup_{|t|<\mu} \|u\|_{C^{1+\alpha}(\Sigma_t)} \leq C_3. \quad (4.9)$$

Thus we have already proved that  $u$  is bounded.



It remains to prove that  $u$  is Hölder continuous in the transversal  $t$  direction. We can prove this by applying a modified one-dimensional Sobolev embedding theorem. This can be done by taking the finite difference of the equation (4.6) with respect to the Besov norms. The proof is exactly the same as before and we refer the reader to the proofs of Lemmas 5.2-5.3 in [CSW]. Thus we conclude that  $u \in C^{\epsilon'}(M)$  for some sufficiently small  $\epsilon' < \epsilon$ .  $\square$

*Proof of the theorem.* Using Lemma 4.1 and (4.5), we have that the curvature form  $\sqrt{-1}\tilde{\Theta}_b^N$  is positive definite on each holomorphic leaf of the Levi-flat hypersurface  $M$  (see Proposition A.2 in the Appendix in [CSW]). Let  $h = 2\text{Im}u$ , where  $u$  is the function obtained in Proposition 4.2. We have

$$\sqrt{-1}\partial_b\bar{\partial}_bh = \sqrt{-1}\tilde{\Theta}_b^N > 0 \quad \text{on } T^{(1,0)}(M) \oplus T^{(0,1)}(M). \quad (4.10)$$

Since  $h$  is continuous on the compact hypersurface  $M$ , it attains its maximum at some point  $p$  in  $M$ . Since  $p$  lies in the interior of some leaf, one obtains a contradiction from (4.10) and the Maximum Principle. This completes the proof of the theorem.  $\square$

## 5. The case for $\mathbb{CP}^2$

To prove the nonexistence of Levi-flat hypersurfaces in  $\mathbb{CP}^2$ , we can study the  $\bar{\partial}$ -Cauchy problem for the top degree forms. There are major differences for compatibility conditions for  $\bar{\partial}$ -closed extensions of  $(0, q)$ -forms when  $q < n - 1$  and  $q = n - 1$ . In general, the space of harmonic  $(p, n - 1)$ -forms on a pseudoconcave domain in  $\mathbb{CP}^n$  is infinite dimensional (see Theorem 3.1 in Hörmander [Hö2]).

For  $q = n - 1$ , there is an additional compatibility condition for the  $\bar{\partial}$ -closed extension of  $(p, n - 1)$ -forms.

**Proposition 5.1.** *Let  $\Omega \subset\subset \mathbb{CP}^n$  be a pseudoconvex domain with  $C^2$  boundary,  $n \geq 2$ , and let  $\Omega^+ = \mathbb{CP}^n \setminus \bar{\Omega}$ . For any  $\bar{\partial}$ -closed  $f \in W_{(p, n-1)}^1(\Omega^+)$ , where  $0 \leq p \leq n$ , the following conditions are equivalent:*

- (1) *There exists  $F \in L_{(p, n-1)}^2(\mathbb{CP}^n)$  such that  $F|_{\Omega^+} = f$  and  $\bar{\partial}F = 0$  in  $\mathbb{CP}^n$  in the distribution sense.*
- (2) *The restriction of  $f$  to  $b\Omega$  satisfies the compatibility condition*

$$\int_{b\Omega} f \wedge \phi = 0, \quad \phi \in L_{(n-p, 0)}^2(\Omega) \cap \text{Ker}(\bar{\partial}).$$

- (3) *Any  $W^1$  extension  $\tilde{f} \in W_{(p, n-1)}^1(\mathbb{CP}^n)$  of  $f$  satisfies the compatibility condition*

$$\int_{\Omega} \bar{\partial}\tilde{f} \wedge \phi = 0, \quad \phi \in L_{(n-p, 0)}^2(\Omega) \cap \text{Ker}(\bar{\partial}).$$

*When  $p \neq n - 1$ , the above conditions are equivalent to*

- (4) *There exists  $u \in W_{(p, n-2)}^1(\Omega^+)$  satisfying  $\bar{\partial}u = f$  in  $\Omega^+$ .*

We remark that any  $f$  in  $W^1(\Omega^+)$  has a trace in  $W^{\frac{1}{2}}(b\Omega^+)$  and any holomorphic  $(n-p, 0)$ -form with  $L^2(\Omega)$  coefficients has a well-defined trace in  $W^{-\frac{1}{2}}(b\Omega)$  (see e.g. [LM]). Thus the pairing between  $f$  and  $\phi$  in (2) is well-defined.

*Proof.* We first show that (1) implies (2).

We assume that there exists a  $\bar{\partial}$ -closed extension  $F$  of a  $\bar{\partial}$ -closed form  $f$ . For  $\phi \in C^1_{(n-p, 0)}(\bar{\Omega}) \cap \text{Ker}(\bar{\partial})$ , by Stokes' theorem, we have

$$0 = \int_{\Omega} \bar{\partial}F \wedge \phi = \int_{\Omega} \bar{\partial}(F \wedge \phi) = \int_{b\Omega} f \wedge \phi = 0.$$

If  $\phi$  is only in  $L^2$ , we use an approximating sequence  $\phi_{\nu} \in C^1_{(n-p, 0)}(\Omega)$  such that  $\phi_{\nu} \rightarrow \phi$  in  $L^2_{(n-p, 0)}(\Omega)$  and  $\bar{\partial}\phi_{\nu} \rightarrow 0$  in  $L^2_{(n-p, 1)}(\Omega)$  by the Friedrichs' Lemma (cf. [CS]). We have

$$\begin{aligned} 0 &= \lim_{\nu \rightarrow \infty} \left( \int_{\Omega} \bar{\partial}F \wedge \phi_{\nu} + (-1)^{p+n-1} \int_{\Omega} F \wedge \bar{\partial}\phi_{\nu} \right) \\ &= \lim_{\nu \rightarrow \infty} \int_{b\Omega} f \wedge \phi_{\nu} = \int_{b\Omega} f \wedge \phi. \end{aligned}$$

To see that (2) implies (3), we observe

$$\int_{\Omega} \bar{\partial}\tilde{f} \wedge \phi = \int_{b\Omega} f \wedge \phi = 0.$$

To show that (3) implies (1), we set

$$T\tilde{f} = -\star\bar{\partial}N_{(n-p, 0)}(\star\bar{\partial}\tilde{f}) \quad \text{on } \Omega.$$

From the proofs of Theorem 1.2 or Corollary 1.6, we have  $\bar{\partial}T\tilde{f} = \bar{\partial}\tilde{f}$  in  $\mathbb{C}P^n$  if we extend  $T\tilde{f}$  to be zero outside  $\Omega$ . Define  $F$  the same as in (2.1). Then  $F \in L^2_{(p, n-1)}(\mathbb{C}P^n)$  and  $F$  is a  $\bar{\partial}$ -closed extension of  $f$ . This proves that conditions (1), (2) and (3) are equivalent.

When  $p \neq n-1$ , the harmonic  $(p, n-1)$ -forms  $\mathcal{H}_{(p, n-1)}(\mathbb{C}P^n) = \{0\}$ . Thus if (1) holds, then there exists  $u \in W^1_{(p, n-2)}(\mathbb{C}P^n)$  satisfying  $\bar{\partial}u = f$  in  $\mathbb{C}P^n$ . Restricting  $u$  to  $\Omega^+$ , we have proved (4). Conversely, if  $f$  is  $\bar{\partial}$ -exact for some  $u \in W^1_{(p, n-2)}(\Omega^+)$ , we can extend  $u$  to be a  $(p, n-2)$ -form in  $W^1_{(p, n-2)}(\mathbb{C}P^n)$ . Then the  $(p, n-1)$ -form  $F = \bar{\partial}u$  is a  $\bar{\partial}$ -closed extension of  $f$  with  $L^2$  coefficients. Thus (1) and (4) are equivalent. The proposition is proved.  $\square$

Proposition 5.1 also holds for any  $\Omega$  with  $C^{1,1}$  Levi-flat boundary. If one can show that any  $\bar{\partial}$ -closed form on  $\Omega^+$  with  $W^1(\Omega^+)$  coefficients extends to be  $\bar{\partial}$ -closed in  $\mathbb{C}P^2$ , i.e., any of the equivalent conditions in Proposition 5.1 holds on a domain with  $C^{1,1}$  Levi-flat boundary, then one can show the nonexistence of  $C^{1,1}$  Levi-flat hypersurfaces in  $\mathbb{C}P^2$  using arguments similar to the proof of the main theorem in Section 4. Notice that in this case, the domain  $\Omega$  is both pseudoconvex and pseudoconcave. But to prove the nonexistence of Levi-flat hypersurfaces in  $\mathbb{C}P^2$ , we need the following  $W^1$  regularity for the  $\bar{\partial}$ -equation.

**Conjecture 1 ( $W^1$  regularity for  $\bar{\partial}$ ).** *Let  $\Omega \subset\subset \mathbb{C}P^2$  be a Lipschitz domain with Levi-flat boundary. For any  $f \in C_{(0,1)}^\infty(\bar{\Omega})$  with  $\bar{\partial}f = 0$ , there exists  $u \in W^1(\Omega)$  such that  $\bar{\partial}u = f$ .*

Conjecture 1 will yield the nonexistence of Lipschitz Levi-flat hypersurfaces in  $\mathbb{C}P^2$ . When  $b\Omega$  is  $C^4$  and Levi-flat, this is proved by Siu (see [Si2]) with  $u \in W^3(\Omega)$ . It seems that one only needs the boundary to be  $C^2$  to have a solution  $u \in W^1(\Omega)$ . Thus we can reduce the smoothness assumption used in [Si2] on  $\Omega$ , but the  $W^1$  regularity of the solution for the  $\bar{\partial}$ -equation cannot be removed.

The following Liouville type result stated in Proposition 4.5 in [CSW] remains open.

**Conjecture 2 (Liouville's Theorem).** *Let  $\Omega^+ \subset\subset \mathbb{C}P^n$  be a pseudoconcave domain with  $C^2$ -smooth boundary (or Lipschitz)  $b\Omega^+$ ,  $n \geq 2$ . Then  $L_{(p,0)}^2(\Omega^+) \cap \text{Ker}(\bar{\partial}) = \{0\}$  for every  $1 \leq p \leq n$  and  $L_{(0,0)}^2(\Omega^+) \cap \text{Ker}(\bar{\partial}) = \mathbb{C}$ .*

This conjecture also implies the nonexistence of Levi-flat hypersurfaces in  $\mathbb{C}P^n$  for  $n \geq 2$ . From Corollary 3.4, the set  $W_{(p,0)}^{1+\epsilon}(\Omega^+) \cap \text{Ker}(\bar{\partial})$  is either zero or constants for Lipschitz pseudoconcave domains. When the boundary is  $C^2$ , this is also true for  $\epsilon = 0$ . Thus it suffices to show that  $W_{(p,0)}^1(\Omega^+) \cap \text{Ker}(\bar{\partial})$  is dense in  $L_{(p,0)}^2(\Omega^+) \cap \text{Ker}(\bar{\partial})$  for the  $C^2$  case. There is still a gap in the the required uniform estimates (4.18) for Proposition 4.5 in [CSW]. We remark that Conjecture 2 is much stronger than the nonexistence of Levi-flat hypersurfaces, since there are many pseudoconcave domains in  $\mathbb{C}P^n$ .

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